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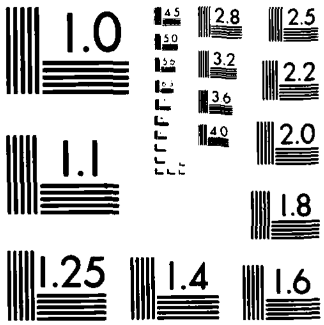

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MULTIPLE CRITICAL POINTS OF INVARIANT FUNCTIONALS AND APPLICATIONS

D. G. Costa and M. Willem

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

This paper deals with some multiplicity results for periodic orbits of Hamiltonian systems and for solution of a non-linear Dirichlet problem. These results follow from an abstract theorem of Lusternik-Schnirelman type as applied to an invariant equation of the form $Lu + \nabla F(u) = 0$ in a Hilbert space $X = L^2(\Omega;\mathbb{R}^N)$, where L is an unbounded self-adjoint operator and F is a C^1 strictly convex function.

delta

L sub 2 (omega, R sub N)

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** Institut de Mathématique Pure et Appliquée - Université Catholique de Louvain, Belgium.

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SIGNIFICANCE AND EXPLANATION

This paper is concerned with existence of multiple solutions of an equation of the form

$$(*) \qquad Lu + \nabla F(u) = 0 \quad ,$$

where L is a self-adjoint operator and F is a strictly convex function. We assume that $\nabla F(0) = F(0) = 0$, so that $u = 0$ is a solution of (*). Loosely speaking, it is reasonable to expect the number of non-trivial solutions of (*) to be related to the number of eigenvalues of the operator $-L$ which are crossed by the function $2F(u)/|u|^2$ as $|u|$ varies from 0 to ∞. We show that under certain conditions this is actually the case. Applications are given to existence of multiple T-periodic solutions of a conservative Hamiltonian system $J\dot{u} + \nabla H(u) = 0$ and to existence of multiple non-radial solutions of the Dirichlet problem for $-\Delta u + g(u) = 0$ in the unit disc of the plane.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

MULTIPLE CRICIAL POINTS OF INVARIANT FUNCTIONALS AND APPLICATIONS

D. G. Costa* and M. Willem**

1. Introduction

This paper is devoted to some multiplicity results for periodic orbits of Hamiltonian systems and for solutions of a non-linear Dirichlet problem. These results follow from an abstract theorem of Lusternik-Schnirelman type, which is a slight (but useful) extension of Ekeland-Lasry's Theorem III.1 in [10].

We first consider the equation

$$(*) \qquad Lu + \nabla F(u) = 0$$

in a Hilbert space $X = L^2(\Omega;\mathbb{R}^N)$, where L is an unbounded self-adjoint operator with no essential spectrum and $F \in C^1(\mathbb{R}^N,\mathbb{R})$ is strictly convex. We assume that $\nabla F(0) = 0$, so that $u = 0$ is a solution of (*). We assume also, without loss of generality, that $F(0) = 0$. Loosely speaking, it seems reasonable to expect the number of *non*-trivial solutions of (*) to be related to the number of eigenvalues of $-L$ crossed by $2F(u)/|u|^2$ as $|u|$ varies from 0 to ∞. As we shall see more precisely in Theorem 2, this heuristic statement actually holds when (*) is equivariant with respect to some group action, so that Lusternik-Schnirelman theory can be used. We apply this theory to the "dual action" introduced by Clarke and Ekeland [7] for Hamiltonian systems. The abstract framework and main results are presented in section 2.

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In section 3, as a first application, we consider the existence of T-periodic solutions of a conservative Hamiltonian system

$$J\dot{u} + \nabla H(u) = 0 \quad ,$$

where $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$ is strictly convex and $u = 0$ is an equilibrium. Using the natural action of $S^1 = \mathbb{R}/T$ provided by the time translations (cf. Fadell-Rabinowitz [12] and Benci [2]), we show that if $\overline{\lim}_{|u|\to\infty} 2H(u)/|u|^2 < 2\pi/T \leq 2\pi j/T < \underline{\lim}_{|u|\to 0} 2H(u)/|u|^2$ for some $j \in \mathbb{N}^*$, then the above Hamiltonian system possesses at least jn non-constant T-periodic solutions describing distinct orbits.

For the non-linear Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

it is classical to use the $\mathbb{Z}_2$-action when g is odd [5]. When Ω is a disc in $\mathbb{R}^2$, the symmetry of the domain was used in [9] instead of the symmetry of the non-linearity. In this case, a natural S^1-action is provided by the rotations. We extend the multiplicity result of [9] to some resonant cases. Moreover the use of the dual action simplifies the proof. It is interesting to note that we obtain, as in [9], non-radial solutions.

Our arguments depend only on the common properties of the usual index theories (cf. [12,2], for example). In particular, for the Dirichlet problem, other symmetries of the domain could be exploited. More general situations and applications to a non-linear string equation will be considered in a subsequent paper.

2. The abstract framework. Main results.

Let X be a Hilbert space on which the group S^1 acts through isometries $S(\theta)$, i.e., for every $\theta \in S^1$, $S(\theta) : X \to X$ is an isometry such that

$$S(\theta_1 + \theta_2) = S(\theta_1)S(\theta_2) \quad ,$$

$$S(0) = \mathrm{Id} \quad ,$$

$$(\theta,u) \mapsto S(\theta)u \text{ is continuous} \quad .$$

We denote by $\mathrm{Fix}(S^1) \subset X$ the subspace of fixed points of X under the S^1-action,

$$\mathrm{Fix}(S^1) = \{u \in X \mid S(\theta)u = u \quad \forall\theta \in S^1\} \quad ,$$

and by ind the cohomological index [12] or the geometrical index [2].

Theorem 1. Let $\phi \in C^1(X,\mathbb{R})$ be an invariant functional bounded from below and satisfying the Palais-Smale condition (PS): every sequence (u_m) such that $\phi(u_m)$ is bounded and $\phi'(u_m) \to 0$ has a convergent subsequence. If $\Omega = \{u \in X \mid \phi(u) < 0\}$ is such that

$$\mathrm{Fix}(S^1) \cap \Omega \cap \{u \in X \mid \phi'(u) = 0\} = \emptyset$$

and if Ω contains a compact invariant set Σ such that

$$\mathrm{ind}\, \Sigma = n \quad ,$$

then Ω contains at least n distinct S^1-orbits of critical points of ϕ.

Proof. It is similar to the one in Ekeland-Lasry [10], with

$$\Gamma_k = \{\gamma \subset \Omega \mid \gamma \text{ is compact, invariant, } \mathrm{ind}\,\gamma \geq k\} \quad ,$$

using also the fact that any compact invariant set which is free of fixed points has a finite index.

■

Remark. Theorem 1 is the S^1-version of a result of Clark [6] for the $\mathbb{Z}_2$-action. But $Fix(\mathbb{Z}_2) = \{0\}$ so that, if ϕ is even, condition

$$Fix(\mathbb{Z}_2) \cap \Omega \cap \{u \in X \mid \phi'(u) = 0\} = \emptyset$$

is equivalent to $\phi(0) > 0$.

The framework to which the above multiplicity theorem will be applied is the following. We consider the equation

$$(*) \qquad Lu + \nabla F(u) = 0$$

in a Hilbert space $X = L^2(\Omega;\mathbb{R}^N)$, where $L : D(L) \subset X \to X$ is an unbounded self-adjoint operator with a discrete pure-point spectrum $\sigma(L) = \{\lambda_i\}$, λ_i of finite multiplicity, and

(1) $F \in C^1(\mathbb{R}^N, R)$ is strictly convex, $F(0) = \nabla F(0) = 0$,

(2) $0 \leq F(u) \leq \gamma \frac{|u|^2}{2} + \alpha$.

The only interesting case is when L is not monotone. So we assume that $\sigma(L) \cap (-\infty,0) \neq \emptyset$ and denote by λ_{-1} the first negative eigenvalue of L.

In the situation described above, it follows that the range of L is closed, $R(L) = \ker(L)^{\perp} \equiv Y$, and the operator $L : D(L) \cap Y \to Y$ has a compact inverse $K : Y \to Y$ with with

$$(i) \qquad (Kv,v)_X \geq \frac{1}{\lambda_{-1}} |v|_X^2$$

for all $v \in Y$. On the other hand, if we also assume

$$(2') \qquad \beta \frac{|u|^2}{2} - \alpha \leq F(u) \leq \gamma \frac{|u|^2}{2} + \alpha, \qquad 0 \leq \alpha,\ 0 < \beta \leq \gamma \ ,$$

then the Legendre-Fenchel transform of F,

$$G(v) = F^*(v) = \sup_u[(v,u) - F(u)] \ ,$$

is a strictly convex C^1 function satisfying

(ii) $$\frac{1}{\gamma}\frac{|v|^2}{2} - \alpha < G(v) < \frac{1}{\beta}\frac{|v|^2}{2} + \alpha \quad .$$

Therefore, we can define the dual action $\phi \in C^1(Y,R)$ by

$$\phi(v) = \frac{1}{2}(Kv,v)_X + \int_\Omega G(v) \quad .$$

<u>Lemma 1</u>. If $v \in Y$ is a critical point of ϕ then there is a solution $u \in D(L)$ of (*) such that $v = -Lu$.

<u>Proof</u>. If v is a critical point of ϕ then

$$(Kv + \nabla G(v), h)_X = 0$$

for all $h \in Y = R(L)$, so that $w = Kv + \nabla G(v) \in \ker(L)$. Letting $u = w - Kv = \nabla G(v)$ we obtain, by duality, $v = \nabla F(u)$. Since $Lu = -v$, it follows that $Lu + \nabla F(u) = 0$.

■

<u>Remark</u>. Related abstract formulations of the Clarke-Ekeland dual action were introduced in [11] and [14].

<u>Lemma 2</u>. If F satisfies (1), (2') with

(3) $$\gamma < -\lambda_{-1} \quad ,$$

then the dual action ϕ

(a) is bounded from below;

(b) satisfies the Palais-Smale condition.

<u>Proof</u>. (a) It follows from (i) and (ii) that

(iii) $$\phi(v) > \frac{1}{2}\left(\frac{1}{\lambda_{-1}} + \frac{1}{\gamma}\right)|v|_X^2 - \alpha|\Omega| \quad ,$$

hence ϕ is bounded from below since $\gamma < -\lambda_{-1}$.

(b) Let $(v_k) \subset Y$ be such that $\phi(v_k)$ is bounded and $\phi'(v_k) \to 0$. Then, by (iii), (v_k) is bounded in X. Going, if necessary, to a subsequence we can assume that $v_k \to v$ weakly in Y. Since K is compact, $Kv_k \to Kv$ in Y. On the other hand, since $\phi'(v_k) \to 0$, we have

$$Kv_k + \nabla G(v_k) - P\nabla G(v_k) = f_k \to 0 \quad \text{in} \quad Y \quad ,$$

where P denotes the orthogonal projection on $\ker(L)$, or, by duality,

$$v_k = \nabla F(-Kv_k + P\nabla G(v_k) + f_k) \quad .$$

Therefore, since $\ker(L)$ is finite dimensional and $\nabla G(v_k)$ is bounded (∇G has linear growth), we can assume, going to a subsequence if necessary, that $P\nabla G(v_k) \to w$ and obtain

$$v_k \to \nabla F(-Kv + w) \quad \text{in} \quad Y \quad ,$$

hence $v_k \to v$ in Y.

■

Lemma 3. Suppose F satisfies (1), (2'),

$$\text{(4)} \qquad \varliminf_{|u|\to 0} \frac{2F(u)}{|u|^2} > -\lambda_{-j} \quad ,$$

where $\lambda_{-j} \in \sigma(L)$, $\lambda_{-j} < \lambda_{-1}$, and

$$\text{(5)} \qquad Z \equiv \ker(L-\lambda_{-1}) \oplus \cdots \oplus \ker(L-\lambda_{-j}) \subset L^{\infty}(\Omega;R^N) \quad .$$

Then there exists $\rho > 0$ such that

$$\phi(v) < 0 \quad \text{for} \quad v \in \Sigma = \{v \in Z| \ |v|_X = \rho\} \quad .$$

Proof. Assumption (4) implies the existence of $\varepsilon > 0$ and $c > -\lambda_{-j}$ such that $F(u) \geq c|u|^2/2$ for $|u| \leq \varepsilon$. On the other hand, there is $\rho' > 0$ such that $|\nabla G(v)| \leq \varepsilon$ for $|v| \leq \rho'$. Since $G(v) = (u,v) - F(u)$ with $u = \nabla G(v)$, we obtain, when $|v| \leq \rho'$,

$$G(v) \leq \max_{|u|\leq\varepsilon} [(u,v) - \frac{c}{2} |u|^2]$$

$$\leq \max_{u} [(u,v) - \frac{c}{2} |u|^2] = \frac{1}{c} \frac{|v|^2}{2} \quad .$$

Now, for $v \in Z$, it is easy to verify the estimate

$$(Kv,v)_X \leq \frac{1}{\lambda_{-j}} |v|_X^2 \quad .$$

Combining these estimates and using (5) we obtain

$$\phi(v) \leq \frac{1}{2} (\frac{1}{\lambda_{-j}} + \frac{1}{c}) |v|_X^2 < 0$$

for $v \in Z$ with $0 < |v|_{L^\infty} \leq \rho'$. The proof is complete since Z is finite dimensional.

■

Remark. It follows from lemma 2 that ϕ has a minimum and from lemma 3 that $\min \phi < 0$. Thus, by lemma 1, under assumptions (1), (2'), (3) - (5), equation (*) admits a non-trivial solution. This result is due to Coron [8]. In order to obtain more non-trivial solutions we shall introduce a group action.

From now on we assume there is an S^1-action on X through isometries $S(\theta)$, $\theta \in S^1$, and that

(6) $\nabla F : X \to X$ and $L : D(L) \subset X \to X$ are equivariant.

(For the unbounded operator L, we mean that $S(\theta)D(L) = D(L)$ and $LS(\theta)u = S(\theta)Lu$ for all $u \in D(L)$, $\theta \in S^1$.)

Then, it is easy to see that $Y = R(L)$ is invariant, $\nabla G : X \to X$ and $K : Y \to Y$ are equivariant and, hence, the dual action ϕ is invariant. We denote by $V = Fix(S^1) \subset X$ the subspace of fixed points of X under the S^1-action,

$$V = \{u \in X \mid S(\theta)u = u \quad \forall\theta \in S^1\} \quad .$$

It is clear that V is an invariant subspace and that $L_0 : D(L) \cap V \to V$, the restriction of L to V, is an equivariant self-adjoint operator with $\sigma(L_0) \subset \sigma(L)$.

<u>Lemma 4</u>. Under assumptions (1), (6) and

(7) if $\lambda_{-\ell} = \sup \sigma(L_0) \cap (-\infty,0) > -\infty$, $(\nabla F(u) - \nabla F(v), u-v) \leq \eta|u-v|^2$ for some $0 < \eta < -\lambda_{-\ell}$,

the only solution of (*) in V is $u = 0$.

<u>Proof</u>. If $\sigma(L_0) \cap (-\infty,0) = \emptyset$ then $L + \nabla F$ is strictly monotone on V and the result follows. So we assume $\sigma(L_0) \cap (-\infty,0) \neq \emptyset$ and denote by $\lambda_{-\ell}$ the first negative eigenvalue of L_0, so that, by (7),

$$(\nabla F(u) - \nabla F(v), u-v) \leq \eta|u-v|^2, \ 0 < \eta < -\lambda_{-\ell} .$$

It follows (cf. Prop. A.5 in [4]) that

$$(\nabla F(u) - \nabla F(v), u-v) \geq \frac{1}{\eta} |\nabla F(u) - \nabla F(v)|^2 .$$

Therefore, if $u \in V$ is a solution of (*), we obtain

$$\frac{1}{\eta}|\nabla F(u)|_X^2 \leq (\nabla F(u),u)_X = (-Lu,u)_X \leq -\frac{1}{\lambda_{-\ell}} |Lu|_X^2 = -\frac{1}{\lambda_{-\ell}} |\nabla F(u)|_X^2 ,$$

and, since $\eta < -\lambda_{-\ell}$, we get $\nabla F(u) = 0$, i.e., $u = 0$, by the strict monotonicity of ∇F.

■

A final assumption we shall make, which is satisfied in most applications, is the following

(8) $K : Y \to L^\infty(\Omega;\mathbb{R}^N)$ is continuous and $\ker(L) \subset L^\infty(\Omega;\mathbb{R}^N)$.

<u>Theorem 2</u>. Under assumptions (1) - (8), there exist at least $n = \text{ind}\,\Sigma$ distinct S^1-orbits of solutions of (*) outside $\text{Fix}(S^1)$. Moreover, $u = 0$ is the only solution of (*) in $\text{Fix}(S^1)$.

__Proof__. We start by showing that $v = 0$ is the only critical point of ϕ in $V = \mathrm{Fix}(S^1)$. Indeed, let $v \in V$ be a critical point of ϕ, so that

$$Kv + \nabla G(v) = w \in \ker(L) \quad .$$

From the equivariance of K and ∇G it follows that $w \in V$, hence $u = w - Kv \in V$. But then lemma 4 implies $u = 0$, i.e., $w = Kv = 0$, so that $v = 0$.

Now, let us first assume (2') instead of (2). Then, lemmas 2, 3 and theorem 1 applied to the dual action ϕ imply the existence of at least $n = \mathrm{ind}\,\Sigma$ distinct orbits $\{L(\theta)v_j \mid \theta \in S^1\}$ of critical points of ϕ. (Note that assumption $\mathrm{Fix}(S^1) \cap \Omega \cap \{v \in Y \mid \phi'(v) = 0\} = \emptyset$ of theorem 1 is automatically satisfied from what we just showed above.) By lemma 1, to each v_j corresponds a solution u_j of (*) such that $v_j = -Lu_j$. If u_j and $u_{j'}$ describe the same orbit then $u_j = S(\theta)u_{j'}$ for some θ, so that $v_j = -Lu_j = -LS(\theta)u_{j'} = S(\theta)(-Lu_{j'}) = S(\theta)v_{j'}$, i.e., v_j and $v_{j'}$ are in the same orbit. But then $j = j'$.

In order to get rid of assumption (2'), we let

$$d = \min\{\frac{1}{2}\,(-\lambda_{-1} - \overline{\lim_{|u|\to\infty}}\,\frac{2F(u)}{|u|^2}),\ \frac{1}{2}\,(-\lambda_{-\ell} - \eta)\} > 0$$

and introduce an increasing convex function $\chi \in C^1(\mathbb{R}^+,\mathbb{R})$ such that

$$\chi(t) = 0 \quad , \qquad \text{if} \quad 0 \leq t \leq R$$

$$\chi(t) = d\,\frac{t^2}{2} \quad , \quad \text{if} \quad 2R \leq t < \infty \quad .$$

Then the function

$$\tilde{F}(u) = F(u) + \chi(|u|)$$

satisfies (1), (2'), (3), (4), (6), (7), so that the equation

$$(\tilde{*}) \qquad\qquad Lu + \nabla\tilde{F}(u) = 0$$

has at least n distinct solutions u_j, $j = 1,\ldots,n$, describing distinct orbits. In order to complete the proof of theorem 2, it suffices to find a bound for $|u_j|_{L^\infty}$ independent of R.

Let $v_j = -Lu_j$ and let $\tilde{\phi}$ be the dual action associated to equation ($\tilde{*}$). It follows from lemma 3 that $\tilde{\phi}(v_j) < 0$. Also, if $\tilde{\gamma}$ is such that

$$d + \overline{\lim_{|u|\to\infty}} \frac{2F(u)}{|u|^2} < \tilde{\gamma} < -\lambda_{-1} \quad ,$$

then

$$\tilde{F}(u) \leq \tilde{\gamma} \frac{|u|^2}{2} + \alpha$$

for some $\alpha \geq 0$ independent of R. We obtain from (iii)

$$0 > \tilde{\phi}(v_j) \geq \frac{1}{2} \left(\frac{1}{\lambda_{-1}} + \frac{1}{\tilde{\gamma}}\right) |v_j|_X^2 - \alpha|\Omega| \quad ,$$

so that

(9) $$|Lu_j|_X^2 = |v_j|_X^2 \leq M$$

for some $M > 0$ independent of R.

On the other hand, by assumption (4), there is $r > 0$ such that $\min_{|u|=r} F(u) > 0$ and so, by the convexity of F, we obtain

$$b|u| - a \leq F(u) \leq \tilde{F}(u)$$

for some $a,b > 0$. Therefore,

$$b|u_j| - a \leq F(u_j) \leq \tilde{F}(u_j) \leq (\nabla\tilde{F}(u_j),u_j) = (-Lu_j,u_j) \quad ,$$

and, after integrating and using (i), we obtain

(10) $$\begin{aligned} b|u_j|_{L^1} &\leq -(Lu_j,u_j)_X + a|\Omega| \leq -\frac{1}{\lambda_{-1}} |Lu_j|_X^2 + a|\Omega| \\ &\leq -\frac{1}{\lambda_{-1}} M + a|\Omega| \quad . \end{aligned}$$

Estimates (9), (10) together with assumption (8) imply a bound for $|u_j|_{L^\infty}$ independent of R, so that the proof of theorem 2 is complete.

■

3. Applications.

We first consider the number of non-constant T-periodic solutions of a Hamiltonian system

$$J\dot{u} + \nabla H(u) = 0 \quad , \tag{11}$$

where $J(x,y) = (-y,x)$. We assume that 0 is an equilibrium, i.e., $\nabla H(0) = 0$, and that $H(0) = 0$.

Theorem 3. Let $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$, $T > 0$ and $j \in \mathbb{N}^*$. If H is strictly convex,

$$\overline{\lim_{|u|\to\infty}} \frac{2H(u)}{|u|^2} < \frac{2\pi}{T} \quad , \tag{12}$$

$$\underline{\lim_{|u|\to 0}} \frac{2H(u)}{|u|^2} > \frac{2j\pi}{T} \quad , \tag{13}$$

then the system (11) has at least jn *non-constant T-periodic solutions* describing distinct orbits.

Proof. Let L be the operator defined by $Lu = J\dot{u}$ with T-periodicity condition on $X = L^2(0,T;\mathbb{R}^{2n})$. Then L is self-adjoint, $\sigma(L) = (2\pi/T)\mathbb{Z}$ and every eigenvalue is of finite multiplicity. Assumption (12) implies (2) and (3) and assumption (13) implies (4) with $\lambda_{-1} = -2\pi/T$, $\lambda_{-j} = -2j\pi/T$ and $F = H$. Since the eigenfunctions are

$$(\cos \frac{2k\pi t}{T})e + (\sin \frac{2k\pi t}{T})Je \quad , \quad k \in \mathbb{Z} \quad ,$$

assumption (5) is satisfied. The group S^1 acts on X through the time translations $S(\theta)$ defined by

$$(S(\theta)v)(t) = v(t+\theta) \quad .$$

It is clear that L and $\nabla F : X \to X$ are equivariant. Moreover, $Fix(S^1)$ is the set of constant functions so that $V = \ker(L)$. Also, it is easy to verify

(8). And, since

$$Z = \ker(L + \frac{2\pi}{T}) \oplus \cdots \oplus \ker(L + \frac{2j\pi}{T}) \quad ,$$

the index of $\Sigma = \{v \in Z \mid |v|_X = \rho\}$ is jn. So, by theorem 2, there exist at least jn distinct S^1-orbits of non-constant solutions of (11) in X.

■

<u>Remark</u>. 1) When $j = 1$ assumptions (12) and (13) imply the existence of a solution with minimal period T [7]. We obtain n T-periodic solutions, but T is not necessarily the minimal period.

2) In general, no more than n distinct orbits with minimal period can be expected.

3) After this work was completed we learned from P. H. Rabinowitz and V. Benci that related multiplicity results were proved by H. Amann-E. Zehnder [1] and V. Benci [3]. We remark that their results were obtained by a different approach under the supplementary assumption the ∇H is linear at 0 and at ∞.

Theorem 2 applies also to Hamiltonians of the form $H(p,q) = |p|^2/2 + V(q)$. We assume as before that $\nabla V(0) = 0$, $V(0) = 0$.

<u>Theorem 4</u>. Let $V \in C^1(\mathbb{R}^n,\mathbb{R})$, $T > 0$ and $j \in \mathbb{N}^*$. If V is strictly convex,

$$\overline{\lim_{|u|\to\infty}} \frac{2V(u)}{|u|^2} < \frac{4\pi^2}{T^2} \quad ,$$

$$\underline{\lim_{|u|\to 0}} \frac{2V(u)}{|u|^2} > \frac{4j^2\pi^2}{T^2} \quad ,$$

then the system

$$\ddot{u} + \nabla V(u) = 0$$

has at least jn non-constant T-periodic solutions describing distinct orbits.

Remarks. 1) The proof of theorem 4 is similar to the proof of theorem 3. It seems that there is no reduction of one result to the other.

2) Related results are contained in [2] but under the assumption that $V''(0)$ exists and that either $V(u)/|u|^2 \to 0$ as $|u| \to \infty$ or ∇V is linear at ∞.

We now consider the non-linear Dirichlet problem on the unit disc Ω in $\mathbb{R}^2$. Let A be the operator $-\Delta$ with Dirichlet condition on $X = L^2(\Omega,\mathbb{R})$. The eigenvalue of A are of the form $\mu = \nu^2$ where ν is a strictly positive zero of some Bessel function J_n, $n \in \mathbb{N}$, of the first kind. The associated eigenfunctions are

$$J_n(\nu r)\cos n\theta,\ J_n(\nu r)\sin n\theta \quad .$$

Note that if ν is a zero of J_0 the $J_0(\nu r)$ is a (radial) eigenfunction associated to $\mu = \nu^2$. Letting $\sigma(A) = \{\mu_1,\mu_2,\ldots\}$, where $0 < \mu_1 < \mu_2 < \ldots$, then each eigenvalue μ_i is either double or simple. (It follows from a deep result of C. Siegel, cf. [13, pg. 485], that the strictly positive zeros of J_{n_1} and J_{n_2} are distinct if $n_1 \neq n_2$.)

Theorem 5. Let $F \in C^1(\mathbb{R},\mathbb{R})$ be a strictly convex function with $F(0) = F'(0) = 0$. Assume that

$$\overline{\lim_{|u|\to\infty}} \frac{2F(u)}{u^2} < \mu_k - \mu_{k-1} \quad , \tag{14}$$

$$\underline{\lim_{|u|\to 0}} \frac{2F(u)}{u^2} > \mu_k - \mu_{k-j} \quad , \tag{15}$$

$$\frac{f(u)-f(v)}{u-v} < \eta < \mu_k - \mu_{k-\ell} \quad , \tag{16}$$

where $k \geq 3$, $\ell-1 \geq j \geq 1$ are such that $\{\mu_{k-\ell+1},\ldots,\mu_{k-1}\} \cap \{\mu>0 \mid J_0(\sqrt{\mu}) = 0\} = \emptyset$, $J_0(\sqrt{\mu_{k-\ell}}) = 0$, and $f = F'$. Then the problem

$$
(17) \qquad \begin{cases} -\Delta u - \mu_k u + f(u) = 0 & \text{in} \quad \Omega \ , \\ u = 0 & \text{on} \quad \partial\Omega \ , \end{cases}
$$

has at least j non-radial geometrically distinct weak solutions. (We say that two function u_1, u_2 are geometrically distinct if after an arbitrary rotation u_1 remains different from u_2.)

Proof. We let L be the operator $A - \mu_k$ with Dirichlet condition on $X = L^2(\Omega;\mathbb{R})$, so that L is self-adjoint and $\sigma(L) = \{\mu_j-\mu_k \mid j = 1,\ldots\}$. Again, assumption (14) implies (2) and (3) and assumption (15) implies (4) with $\lambda_{-1} = \mu_{k-1} - \mu_k$, $\lambda_{-j} = \mu_{k-j} - \mu_k$. Also, assumption (5) is automatically satisfied. We let the group S^1 act on X through the rotations,

$$
(S(\theta)v)(x) = v(R(\theta)x) \quad ,
$$

where $R(\theta)x = R(\theta)(x_1,x_2) = (x_1\cos\theta - x_2\sin\theta,\ x_1\sin\theta + x_2\cos\theta)$. Then it is clear that L and $f = F' : X \to X$ are equivariant and $\mathrm{Fix}\ (S^1)$ is the set of radial functions. Finally, assumption (16) implies (7) with $\lambda_{-\ell} = \mu_{k-\ell} - \mu_k$. And, since

$$
Z = \ker(-\Delta - \mu_{k-1}) \oplus \quad \oplus \ker(-\Delta - \mu_{k-j})
$$

where each summand is two-dimensional, the index of $\Sigma = \{v \in Z \mid |v|_X = \rho\}$ is j. Therefore, theorem 2 implies the existence of at least j non-radial geometrically distinct weak solutions of (17).

■

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This paper deals with some multiplicity results for periodic orbits of Hamiltonian systems and for solution of a non-linear Dirichlet problem. These results follow from an abstract theorem of Lusternik-Schnirelman type as applied to an invariant equation of the form $Lu + \nabla F(u) = 0$ in a Hilbert space $X = L^2(\Omega, \mathbb{R}^N)$, where L is an unbounded self-adjoint operator and F is a C^1 strictly convex function.

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